

\mathbb{Q} -COMPLEMENTS ON LOG SURFACES

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Introduction

In this paper the log surfaces without \mathbb{Q} -complement are classified. In particular, they are non-rational always. This result takes off the restriction in the theory of complements and allows one to apply it in the most wide class of log surfaces (S, D) , where the pair (S, D) is log canonical and the divisor $-(K_S + D)$ is nef. For more information see the papers [5] and [2], especially [5, Theorems 2.3 and 4.1], [2, Theorems 2.1 and 3.1].

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1. Classification theorem

We work over an algebraically closed field k of characteristic zero. The main definitions, notations and notions used in the paper are given in [1], [3].

Definition 1.1. Let (X, D) be a pair, where D is a subboundary. Then a \mathbb{Q} -complement of $K_X + D$ is a log divisor $K_X + D'$ such that $D' \geq D$, $K_X + D'$ is log canonical and $n(K_X + D') \sim 0$ for some $n \in \mathbb{N}$.

Example 1.2. [5, Example 1.1] 1) Let \mathcal{E} be an indecomposable vector bundle of rank two and degree 0 over an elliptic curve Z . Then \mathcal{E} is a nontrivial extension

$$0 \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_Z \longrightarrow 0,$$

see [4]. Consider the ruled surface $X = \mathbb{P}_Z(\mathcal{E})$. Let C be the unique section corresponding to the exact sequence. Note that $C|_C \sim \det \mathcal{E} \sim 0$, $-K_X \sim 2C$ [4]. In particular, $-K_X$ is nef and Mori cone $\overline{\text{NE}}(X)$ is generated by two rays $R_1 = \mathbb{R}_+[C]$ and $R_2 = \mathbb{R}_+[f]$, where f is a fiber.

We claim that C is the unique curve in R_1 . In fact, let there is a curve $L \neq C$ in R_1 then $L \equiv mC$, where $m = L \cdot f \geq 2$. It is

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easy to prove that $L \sim mC$. Therefore the linear system $|L|$ gives a structure of an elliptic fibration on X with a multiple fiber C . Hence $C|_C$ is an m -torsion element in $\text{Pic}(C)$, a contradiction with $C|_C \sim 0$. We proved that the log surface (X, cC) does not have \mathbb{Q} -complement, where $0 \leq c \leq 1$.

2) Consider the pair (X, C) from the previous example. Let P_1, \dots, P_r be the arbitrary points of C . Take any number of blow-ups at P_1, \dots, P_r . We obtain the pair (\tilde{X}, \tilde{C}) , where $K_{\tilde{X}} + \tilde{C} = g^*(K_X + C)$ and \tilde{C} is a proper transform of C . It is clear that (\tilde{X}, \tilde{C}) does not have \mathbb{Q} -complement. If we contract any chains of (-2) -curves on \tilde{X} and maybe \tilde{C} (if it is possible) then the log surface obtained does not have \mathbb{Q} -complement also.

It is obvious that all these log surfaces do not have complement too (see the definition of complement in [3, Definition 4.1.3]).

Theorem 1.3. *Let S be a normal projective surface and D be a boundary on S such that $K_S + D$ is log canonical and $-(K_S + D)$ is nef. Assume that (S, D) does not have \mathbb{Q} -complement. Then the pair (S, D) is of example 1.2, in particular S is non-rational.*

Proof. Let $f: \tilde{S} \rightarrow S$ be a minimal resolution and $K_{\tilde{S}} + \tilde{D} = f^*(K_S + D)$. Then the pair (\tilde{S}, \tilde{D}) does not have \mathbb{Q} -complement too. By abundance theorem [1, Theorem 8.5] the kodaira dimension $k(\tilde{S}) = -\infty$. Consider two cases.

A) Let \tilde{S} be a rational surface. Since $\tilde{S} \not\cong \mathbb{P}^2, \mathbb{F}_0$ then some model of \tilde{S} is \mathbb{F}_n ($n \geq 1$). We have $g: \tilde{S} \rightarrow \mathbb{F}_n \rightarrow Z \cong \mathbb{P}^1$. Let \tilde{E}_∞ be the proper transform of the minimal section of \mathbb{F}_n .

Now we construct the divisor $\tilde{D}' \geq \tilde{D}$ such that $K_{\tilde{S}} + \tilde{D}' \equiv 0/Z$, $(K_{\tilde{S}} + \tilde{D}') \cdot \tilde{E}_\infty = 0$ and the pair (\tilde{S}, \tilde{D}') is log canonical. Hence $K_{\tilde{S}} + \tilde{D}' \equiv 0$ and $K_{\tilde{S}} + \tilde{D}'$ is a \mathbb{Q} -complement of $K_{\tilde{S}} + \tilde{D}$ by abundance theorem, a contradiction.

Let f_1, \dots, f_k be the reducible fibers of g . Let f'_1, \dots, f'_k be any irreducible components of f_1, \dots, f_k . By considering the linear system $|mE_0|$ on \mathbb{F}_n , where E_0 is a zero section and $m \gg 0$ we obtain a free pencil $|L|$ on \tilde{S} such that $L \cdot \tilde{E}_\infty = 0$ and $f_i \cap L_{\text{gen}} \subset f'_i$ are the generic points of f'_i for the general element $L_{\text{gen}} \in |L|$. Hence adding the required number of different L_{gen} and general fibers of g we obtain \tilde{D}' .

B) Let \tilde{S} be a non-rational surface. Then we have a contraction onto a curve $f: \tilde{S} \rightarrow Z$, where general fiber is \mathbb{P}^1 and $p_a(Z) \geq 1$. By [3, Lemma 8.2.2, Corollary 8.2.3] no components of $\text{Supp } \tilde{D}$ are contained

in the fibers of f and the pair (\tilde{S}, \tilde{D}) is canonical. There are two variants.

1) Let $\tilde{D} = \tilde{C} + \tilde{D}_1$, where \tilde{C} is an irreducible curve. Hence $p_a(Z) = 1$ and we have a birational contraction $\tilde{S} \rightarrow \bar{S}$ such that $K_{\bar{S}} + \tilde{D} \equiv 0/\bar{S}$, where \bar{S} is a minimal model of \tilde{S} . Therefore the pair (\bar{S}, \bar{D}) is without \mathbb{Q} -complement, where \bar{D} is the image of \tilde{D} . By [5, Corollary 2.2] and [4] the surface \bar{S} is of example 1.2 1) and $\bar{D} = \bar{C}$. Q.E.D.

2) Let $\tilde{D} = \sum d_i \tilde{D}_i$, where $d_i < 1$ and let \bar{S} be a minimal model of \tilde{S} . By [2, Proof of theorem 3.1] we have $E^2 \geq 0$ for every curve E on \bar{S} , $p_a(Z) = p_a(\bar{D}_i) = 1$, $K_{\bar{S}} \cdot \bar{D}_i = \bar{D}_i^2 = \bar{D}_i \cdot \bar{D}_j = 0$ for all i, j . It follows easily that we can consider $\tilde{D}' = \tilde{D}_1 + \sum_{i \geq 2} d_i \tilde{D}_i$ instead of \tilde{D} by [4] (if $\tilde{D} = 0$ then $\tilde{D}' = \tilde{C}$, where \tilde{C} is a proper transform of an irreducible curve \bar{C} such that $K_{\bar{S}} \cdot \bar{C} = \bar{C}^2 = 0$). The problem is reduced to the variant 1). \square

Remark 1.4. The log surface without complement is the same one. This fact is proved similarly. Thus the log surfaces with complement are equivalent to the log surfaces with \mathbb{Q} -complement.

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